

# NEIGHBOUR TRANSITIVITY ON CODES IN HAMMING GRAPHS

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**ABSTRACT.** We consider a *code* to be a subset of the vertex set of a *Hamming graph*. In this setting a *neighbour* of the code is a vertex which differs in exactly one entry from some codeword. This paper examines codes with the property that some group of automorphisms acts transitively on the *set of neighbours* of the code. We call these codes *neighbour transitive*. We obtain sufficient conditions for a neighbour transitive group to fix the code setwise. Moreover, we construct an infinite family of neighbour transitive codes, with *minimum distance*  $\delta = 4$ , where this is not the case. That is to say, knowledge of even the complete set of code neighbours does not determine the code.

## 1. INTRODUCTION

We consider codes to be subsets of ordered  $m$ -tuples from a fixed alphabet  $Q$  of size  $q$  and so it is natural to consider codes as subsets of vertices of *Hamming graphs* (see Section 2). In this setting a codeword in which exactly one of the entries has been changed is adjacent in the Hamming graph to the codeword and, provided it is not a codeword itself, we call it a *neighbour* of that codeword in the code. For a code  $C$  in a Hamming graph,  $\Gamma = H(m, q)$ , the *set of neighbours of  $C$*  is the subset  $\Gamma_1(C)$  consisting of the vertices of  $\Gamma$  which are not in  $C$  but are joined by an edge to at least one element of  $C$ . In this paper we examine codes with the following property.

**Definition 1.1.** Let  $C$  be a code in  $H(m, q)$ . Then we say that  $C$  is *neighbour transitive* if there exists a subgroup  $X$  of the automorphism group of  $H(m, q)$  that fixes setwise and acts transitively on the set of neighbours of  $C$ . If we want to specify the group, we say  $C$  is  $X$ -neighbour transitive.

Some much stronger group theoretic conditions than those introduced here, such as *complete transitivity* have been studied previously in [11], then in [1], [2] and in a more general context in [6].

In Definition 1.1 it is not assumed that  $X$  acts transitively on  $C$ . Indeed it was the question of whether the neighbour transitive group  $X$  was forced to fix  $C$  setwise that led to the study and the results of this paper. The answer depends on the *minimum distance*,  $\delta$ , of the code  $C$ . This is the minimum number of positions in which two distinct codewords from  $C$  differ; equivalently  $\delta$  is the minimum distance in  $H(m, q)$  between distinct codewords in  $C$ .

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**Theorem 1.2.** *Let  $C$  be a code in  $\Gamma = H(m, q)$  with  $\delta \geq 3$ , and let  $x$  be an automorphism of  $\Gamma$  fixing setwise the set  $\Gamma_1(C)$  of neighbours of  $C$ . Then at least one of the following holds:*

- (1)  *$x$  fixes  $C$  setwise, or*
- (2)  *$\delta = 4$ ,  $q = 2$  and  $m$  is even,*
- (3)  *$\delta = 3$ , and  $m(q - 1)$  is even.*

*Moreover, for each even  $m$  at least 4, there exists an  $X$ -neighbour transitive code  $C \subset H(m, 2)$  with minimum distance  $\delta = 4$  such that  $X$  does not fix  $C$  setwise.*

In Section 2 we introduce our notation, and provide some interesting facts about codes in Hamming graphs. In Section 3 we introduce *pre-codewords*. Pre-codewords are useful in proving our main assertions, which is done in Section 4. In Section 5 we introduce a family of codes with  $\delta = 4$  and  $m$  even, that have the property that for each code in this family, the setwise stabiliser of the neighbour set in the automorphism group of the Hamming graph does not fix the code setwise. Moreover, we prove that each code in this family is neighbour transitive.

**1.1. Relevance to Error Correction for Codes.** Error correcting codes are used to maintain the integrity of data across noisy communication channels and in storage systems. The standard scenario involves a message, or data, being transmitted over a noisy communication channel. The noise that is added during transmission may result in errors occurring and the message changing. Error correcting codes are used to encode the message before transmission by adding a certain amount of redundancy so that the likelihood of recovering the original message is increased. For a broad examination of error correcting codes see [8], [9] and [10].

An assumption frequently made in decoding procedures for error correcting codes is that the probability of a transmission error is independent both of the position in which the error occurs, and also of the incorrect letter of the alphabet occurring in that position, see [9, p.4] and [10, p.5]. In other words, the probability of each error occurring is equally likely. The concept of a code being neighbour transitive is a group theoretic analogue of this assumption.

Studying codes that are neighbour transitive has led to unexpected new constructions of codes with large minimum distance, along with their automorphism groups [4, Chapter 5], as well as new classifications of some families of *completely regular codes*, see [5].

## 2. NOTATION

In this section we introduce the Hamming graphs and automorphisms of the Hamming graph, and some notation that allows us to work with codes and their neighbours.

**2.1. Hamming Graphs.** The Hamming graph with parameters  $m, q$ , has a vertex set which consists of  $m$ -tuples with entries from a set  $Q$  of size  $q$ . An edge exists between two vertices if and only if they differ in precisely one entry. We denote the Hamming graph by  $\Gamma = H(m, q)$ . In the Hamming graph, the

*Hamming distance* between two vertices is defined to be the number of entries in which the two vertices differ. We use  $d(\alpha, \beta)$  to denote the distance between the vertices  $\alpha$  and  $\beta$ .

Sometimes we want to make reference to a particular vertex in  $H(m, q)$ . In order to do this we let  $0$  be a distinguished element of the alphabet. This allows us to consider the zero vertex,  $(0, \dots, 0) = \mathbf{0}$ . Let  $\beta$  be a vertex of  $H(m, q)$ . Then we define the *weight* of  $\beta$  to be the number of non-zero entries of  $\beta$ , which we denote by  $\text{wt}(\beta)$ . This is equal to the distance between  $\mathbf{0}$  and  $\beta$  in  $H(m, q)$ .

The automorphism group of the Hamming graph  $H(m, q)$  is the semi-direct product  $N \rtimes L$  where  $N \cong S_q^m$  and  $L \cong S_m$ , see [3, Theorem 9.2.1]. Throughout this paper we denote this group by  $G$ , and for a subset  $S$  of vertices in  $H(m, q)$  we let  $G_S$  denote the setwise stabiliser in  $G$  of  $S$ . Let  $g = (g_1, \dots, g_m) \in N$ ,  $\sigma \in L$  and  $\alpha = (\alpha_1, \dots, \alpha_m) \in H(m, q)$ . Then  $g$  and  $\sigma$  act on  $\alpha$  in the following way:

$$\begin{aligned}\alpha^g &= (\alpha_1^{g_1}, \dots, \alpha_m^{g_m}) \\ \alpha^\sigma &= (\alpha_{1\sigma^{-1}}, \dots, \alpha_{m\sigma^{-1}}).\end{aligned}$$

It is straight forward to show that  $G$  acts transitively on the vertex set of  $H(m, q)$ .

Let  $\beta$  be a vertex in the Hamming graph. We define the *neighbours* of  $\beta$  to be the set

$$\Gamma_1(\beta) = \{\gamma \in H(m, q) \mid d(\beta, \gamma) = 1\}.$$

We have the following general result about vertices of  $H(m, q)$ .

**Lemma 2.1.** *Let  $\alpha$  and  $\beta$  be distinct vertices in  $H(m, q)$ . Suppose that  $d(\alpha, \beta) = 2$ . Then  $|\Gamma_1(\alpha) \cap \Gamma_1(\beta)| = 2$ .*

*Proof.* Since  $G$  acts transitively on  $H(m, q)$ , we can assume without loss of generality that  $\alpha = \mathbf{0}$ . Therefore  $\Gamma_1(\alpha)$  consists of all  $m(q-1)$  weight one vertices. As  $d(\alpha, \beta) = 2$ , it follows that  $\beta$  has weight 2. Thus  $\Gamma_1(\beta)$  consists of  $(m-2)(q-1)$  vertices of weight 3,  $2(q-2)$  vertices of weight 2 and exactly 2 vertices of weight 1. Thus  $|\Gamma_1(\alpha) \cap \Gamma_1(\beta)| = 2$ .  $\square$

Let  $\text{Triples} = \{(\alpha, \nu, \beta) \mid d(\alpha, \beta) = 2 \text{ and } \nu \in \Gamma_1(\alpha) \cap \Gamma_1(\beta)\}$ . Let  $y \in G$  act on  $(\alpha, \nu, \beta) \in \text{Triples}$  as follows:  $(\alpha, \nu, \beta)^y = (\alpha^y, \nu^y, \beta^y)$ .

**Lemma 2.2.**  *$G$  acts transitively on the set Triples.*

*Proof.* Let 0 and 1 be distinguished elements of  $Q$ , and let  $(\alpha, \nu, \beta)$  be an element of Triples. Since  $G$  acts transitively on the vertices of  $H(m, q)$ , an arbitrary triple can be mapped to one with first entry  $\mathbf{0}$ , so we can assume that  $\alpha = \mathbf{0}$ . Therefore we can also assume that  $\nu$  and  $\beta$  have weight 1 and 2 respectively. Moreover, since  $\nu \in \Gamma_1(\alpha) \cap \Gamma_1(\beta)$ ,  $\beta$  has a non-zero entry in common with  $\nu$ , that is

common in entry  $i$  position and in the element  $a_i$  of  $Q$ . Thus

$$\begin{aligned}\alpha &= (0, \dots, 0) \\ \nu &= (0, \dots, a_i, \dots, 0) \\ \beta &= (0, \dots, a_i, \dots, a_j, \dots, 0).\end{aligned}$$

The stabiliser of  $\alpha$  in  $G$  is isomorphic to the wreath product  $S_{q-1} \wr S_m$ . Consider a group element  $g\sigma = (g_1, \dots, g_m)\sigma \in G_\alpha$ , where  $a_i^{g_i} = 1$ ,  $a_j^{g_j} = 1$ , and  $i^\sigma = 1$ ,  $j^\sigma = 2$ . Then

$$(\alpha, \nu, \beta)^{g\sigma} = ((0, \dots, 0), (1, 0, \dots, 0), (1, 1, 0, \dots, 0)).$$

As we can map any element of *Triples* to this particular element, it follows that  $G$  acts transitively on *Triples*.  $\square$

**2.2. Codes in  $H(m, q)$ .** As mentioned before, we consider codes in  $H(m, q)$  to be subsets of the vertex set. Let  $C$  be a code in  $H(m, q)$ . We define the *minimum distance of  $C$*  to be

$$\delta = \min\{d(\alpha, \beta) \mid \alpha, \beta \in C, \alpha \neq \beta\}.$$

We define the *set of neighbours of  $C$*  to be the set  $\Gamma_1(C) = (\cup_{\alpha \in C} \Gamma_1(\alpha)) \setminus C$ . Observe that if  $\delta \geq 2$  then  $\Gamma_1(C) = \cup_{\alpha \in C} \Gamma_1(\alpha)$ , and if  $\delta \geq 3$ , this is a disjoint union.

Recall  $G = \text{Aut}(\Gamma)$ . We define the *automorphism group of  $C$*  to be the setwise stabiliser of  $C$  in  $G$ , which we denote by  $\text{Aut}(C)$ . Traditionally coding theorists regard certain weight preserving subgroups of  $\text{Aut}(C)$  as the automorphism group of  $C$  (see [7, Sec. 1.6-1.7] for a nice explanation). However, because we are interested in groups of automorphisms acting transitively on neighbour sets of codes, which may contain vertices of different weights, we use this more general notion of  $\text{Aut}(C)$ . The following result shows that  $\Gamma_1(C)$  is necessarily  $\text{Aut}(C)$ -invariant.

**Lemma 2.3.** *Let  $C$  be a code in  $H(m, q)$ . Then  $\text{Aut}(C) \leq G_{\Gamma_1(C)}$ .*

*Proof.* Let  $\nu \in \Gamma_1(C)$ . Then there exists  $\alpha \in C$  such that  $d(\nu, \alpha) = 1$ . Let  $x \in \text{Aut}(C)$ . Because adjacency is preserved by automorphisms it follows that  $d(\nu^x, \alpha^x) = 1$ , and so  $d(\nu^x, C) \leq 1$ . Suppose  $\nu^x \in C$ . Then because  $x \in \text{Aut}(C)$  it follows that  $\nu = (\nu^x)^{x^{-1}} \in C$ , which is a contradiction. Thus  $\nu^x \in \Gamma_1(C)$ .  $\square$

Suppose that  $C$  is  $X$ -neighbour transitive (see Definition 1.1). Since  $X$  and  $\text{Aut}(C)$  leave  $\Gamma_1(C)$  invariant, we have that  $C$  is also  $\langle X, \text{Aut}(C) \rangle$ -neighbour transitive, where  $\langle X, \text{Aut}(C) \rangle$  is the group generated by  $X$  and  $\text{Aut}(C)$ . Therefore we may assume that  $\text{Aut}(C) \leq X$ . The question that this paper addresses is: *when does  $\text{Aut}(C) = X$ ?*

We are interested in distinguishing between codes which are inherently different. Therefore we introduce the following concept.

**Definition 2.4.** Let  $C$  and  $C'$  be two codes in  $H(m, q)$ . We say that  $C$  and  $C'$  are *equivalent* if there exists an automorphism  $y$  of  $H(m, q)$  such that  $C^y = C'$ .

Equivalence preserves several important properties.

**Lemma 2.5.** *Let  $C$  be a code in  $H(m, q)$  with minimum distance  $\delta$ , and let  $y \in G$ . Then  $C^y$  has minimum distance  $\delta$  and  $\text{Aut}(C^y) = y^{-1}\text{Aut}(C)y$ . Moreover, if  $C$  is  $X$ -neighbour transitive then  $C^y$  is  $(y^{-1}Xy)$ -neighbour transitive.*

*Proof.* As automorphisms preserve distance in  $H(m, q)$ , it follows that if  $C$  has minimum distance  $\delta$ , so too does  $C^y$ . It is straight forward to prove that  $y^{-1}\text{Aut}(C)y = \text{Aut}(C^y)$ . Now suppose that  $C$  is  $X$ -neighbour transitive. It is clear that  $y^{-1}Xy$  fixes the neighbours of  $C^y$  setwise. Let  $\nu_1$  and  $\nu_2$  be neighbours of  $\alpha_1$  and  $\alpha_2$ , respectively, in  $C^y$ . Then  $\nu_1^{y^{-1}}$  and  $\nu_2^{y^{-1}}$  are neighbours of  $\alpha_1^{y^{-1}}$  and  $\alpha_2^{y^{-1}}$ , respectively, in  $C$ . Since  $C$  is  $X$ -neighbour transitive, there exists  $x \in X$  such that  $\nu_1^{y^{-1}x} = \nu_2^{y^{-1}}$ , and so  $\nu_1^{y^{-1}xy} = \nu_2$ .  $\square$

### 3. PRE-CODEWORDS

Let  $C$  be a code in  $H(m, q)$ . The main investigation of this paper is to determine when the setwise stabiliser in  $G$  of  $\Gamma_1(C)$  fixes  $C$  setwise. In this section we introduce the concept of a *pre-codeword* which enables us to examine this question further. Firstly we consider the case where the setwise stabiliser of the neighbours does not fix the code setwise.

**Lemma 3.1.** *Let  $C$  be a code with  $\delta \geq 3$ . Suppose there exists  $\alpha \in C$  and  $y \in G_{\Gamma_1(C)}$  such that  $\alpha^y \notin C$ . Then for all  $\nu \in \Gamma_1(\alpha)$ , there exists a unique vertex  $\pi \in \Gamma_2(\alpha)$  such that  $\pi^y \in C$  and  $\nu \in \Gamma_1(\pi)$ .*

*Proof.* Note that  $\Gamma_1(C) = \cup_{\beta \in C} \Gamma_1(\beta)$  since  $\delta \geq 3$ . Let  $\nu \in \Gamma_1(\alpha)$ . Since  $y \in G_{\Gamma_1(C)}$  it follows that  $\nu^y \in \Gamma_1(C)$ . Hence there exists  $\beta \in C$  such that  $\nu^y \in \Gamma_1(\beta)$ . Let  $\pi = \beta^{y^{-1}}$ . Then  $\nu \in \Gamma_1(\pi)$ . It follows that  $d(\alpha, \pi) \leq 2$ . Moreover  $\pi \neq \alpha$  since  $\pi^y = \beta \in C$ , while  $\alpha^y \notin C$ . Consequently  $\pi \notin C$  as  $\delta \geq 3$ .

Now suppose that  $d(\alpha, \pi) = 1$ . Then  $1 = d(\alpha^y, \pi^y) = d(\alpha^y, \beta)$  and since  $\beta \in C$  and  $\alpha^y \notin C$  this implies that  $\alpha^y \in \Gamma_1(C)$ . Since  $y$  fixes  $\Gamma_1(C)$  setwise it follows that  $\alpha \in \Gamma_1(C)$  which is a contradiction since  $\alpha \in C$ . Thus  $\pi \in \Gamma_2(\alpha)$ , and so  $\pi$  has all the required properties. Suppose there exists  $\pi' \neq \pi$  with  $\pi' \in \Gamma_2(\alpha)$  such that  $\nu \in \Gamma_1(\pi')$  and  $\pi'^y \in C$ . Then  $d(\pi^y, \pi'^y) \leq 2$  contradicting the fact that  $\delta \geq 3$ .  $\square$

Thus for every neighbour  $\nu$  of  $\alpha$  there exists a unique vertex  $\pi \notin C$  adjacent to  $\nu$  which gets mapped into  $C$  by  $y$ . However, these vertices depend on  $\alpha$  and the particular  $y$ . We now introduce the concept of a *pre-codeword*.

**Definition 3.2.** Let  $C$  be a code with  $\delta \geq 3$ . Let  $\alpha$  be a codeword and  $y \in G_{\Gamma_1(C)}$  such that  $\alpha^y \notin C$ . Then a *pre-codeword* of  $\alpha$  with respect to  $y$  is a vertex  $\pi$  such that  $d(\alpha, \pi) = 2$  and  $\pi^y \in C$ . We denote the set of all pre-codewords of  $\alpha$  with respect to  $y$  by  $\text{Pre}(\alpha, y)$ .

By definition each pre-codeword of  $\alpha$  with respect to  $y$  is at distance two from  $\alpha$ . We now use properties of Hamming graphs to determine the cardinality of  $\text{Pre}(\alpha, y)$ .

**Lemma 3.3.** *Let  $C$  be a code with  $\delta \geq 3$  and let  $\alpha \in C$  and  $y \in G_{\Gamma_1(C)}$  such that  $\alpha^y \notin C$ . Then*

- (i)  $\{\Gamma_1(\alpha) \cap \Gamma_1(\pi) \mid \pi \in \text{Pre}(\alpha, y)\}$  forms a partition of  $\Gamma_1(\alpha)$ .
- (ii)  $|\text{Pre}(\alpha, y)| = m(q-1)/2$ , in particular  $m(q-1)$  is even; and
- (iii) for each  $\pi \in \text{Pre}(\alpha, y)$ ,  $\Gamma_1(\pi) \subset \Gamma_1(C)$ .

*Proof.* Let  $\nu \in \Gamma_1(\alpha)$ . By Lemma 3.1 there exists a unique vertex  $\pi \in \text{Pre}(\alpha, y)$  such that  $\nu \in \Gamma_1(\pi)$ . Thus  $\bigcup_{\pi \in \text{Pre}(\alpha, y)} \Gamma_1(\alpha) \cap \Gamma_1(\pi)$  covers  $\Gamma_1(\alpha)$ , and each  $\nu$  lies in a unique subset  $\Gamma_1(\alpha) \cap \Gamma_1(\pi)$ . Hence  $\{\Gamma_1(\alpha) \cap \Gamma_1(\pi) \mid \pi \in \text{Pre}(\alpha, y)\}$  is a partition of  $\Gamma_1(\alpha)$  and (i) holds. Furthermore, by Lemma 2.1, each cell of this partition comprises exactly two neighbours of  $\alpha$ . Thus  $2 \times |\text{Pre}(\alpha, y)| = |\Gamma_1(\alpha)| = m(q-1)$ , giving us (ii). Let  $\pi \in \text{Pre}(\alpha, y)$  and suppose  $\nu$  is a neighbour of  $\pi$ . Then  $\nu^y$  is a neighbour of the codeword  $\pi^y$ , that is  $\nu^y \in \Gamma_1(C)$ . As  $y$  stabilises  $\Gamma_1(C)$  setwise, we have that  $\nu \in \Gamma_1(C)$  and (iii) follows.  $\square$

Let  $\pi \in \text{Pre}(\alpha, y)$ . Consider the set  $C(\pi) = \{\beta \in C \mid (\beta, \nu, \pi) \in \text{Triples for some } \nu \in \Gamma_1(\pi)\}$ . We now demonstrate some similar results for  $C(\pi)$  to those for  $\text{Pre}(\alpha, y)$ .

**Lemma 3.4.** *Let  $\delta$ ,  $\alpha$  and  $y$  be as in Definition 3.2 and let  $\pi \in \text{Pre}(\alpha, y)$ , and  $C(\pi)$  as above. Then*

- (i)  $C(\pi) = \Gamma_2(\pi) \cap C$ ,
- (ii)  $\{\Gamma_1(\beta) \cap \Gamma_1(\pi) \mid \beta \in C(\pi)\}$  is a partition of  $\Gamma_1(\pi)$ ,
- (iii)  $|C(\pi)| = m(q-1)/2$ , in particular  $m(q-1)$  is even, and
- (iv)  $\beta^y \notin C$  for all  $\beta \in C(\pi)$ .

*Proof.* Let  $\beta \in C(\pi)$ , thus  $\beta \in C$ . By the definition of **Triples**,  $\beta \in \Gamma_2(\pi)$ . Thus  $C(\pi) \subseteq \Gamma_2(\pi) \cap C$ . Conversely if  $\beta' \in \Gamma_2(\pi) \cap C$ , then there exists  $\nu \in \Gamma_1(\pi) \cap \Gamma_1(\beta')$  by Lemma 2.1 and so  $(\beta', \nu, \pi) \in \text{Triples}$ . Thus  $\beta' \in C(\pi)$ . Therefore (i) holds. From the definition of  $C(\pi)$ ,  $\bigcup_{\beta \in C(\pi)} (\Gamma_1(\beta) \cap \Gamma_1(\pi)) = \Gamma_1(\pi)$ . If  $\nu \in \Gamma_1(\beta) \cap \Gamma_1(\beta') \cap \Gamma_1(\pi)$  for  $\beta, \beta' \in C(\pi)$ , then  $d(\beta, \beta') \leq 2$  and since  $\delta \geq 3$  it follows that  $\beta = \beta'$ . Hence  $\{\Gamma_1(\beta) \cap \Gamma_1(\pi) \mid \beta \in C(\pi)\}$  is a partition of  $\Gamma_1(\pi)$ , and (ii) holds. Furthermore, by Lemma 2.1, each cell of the partition comprises of exactly two neighbours of  $\pi$ . Thus  $2 \times |C(\pi)| = |\Gamma_1(\pi)| = m(q-1)$ , and (iii) follows. In particular  $m(q-1)$  is even. Let  $\beta \in C(\pi)$ . As  $d(\pi^y, \beta^y) = 2$  and  $\delta \geq 3$ , we can conclude that  $\beta^y \notin C$ , so (iv) holds.  $\square$

#### 4. MAIN RESULTS

In this section we use the results from Section 3 to find sufficient conditions under which  $G_{\Gamma_1(C)}$  fixes  $C$  setwise. Firstly we consider the case where  $C$  has a large minimum distance.

**Lemma 4.1.** *Let  $C$  be a code with  $\delta \geq 5$ . Then  $C$  is fixed setwise by  $G_{\Gamma_1(C)}$ .*

*Proof.* Since  $\delta \geq 5$  it follows that  $m \geq 5$ . Suppose  $C$  is not fixed setwise by  $G_{\Gamma_1(C)}$ . Then by Lemma 3.3 there exist  $\alpha \in C$  and  $y \in G_{\Gamma_1(C)}$  such that  $|\text{Pre}(\alpha, y)| \geq 3$ . Let  $\pi_1$  and  $\pi_2$  be distinct elements of  $\text{Pre}(\alpha, y)$ . Therefore  $\pi_1, \pi_2 \in \Gamma_2(\alpha)$  and  $\pi_1^y, \pi_2^y \in C$ . It follows that  $d(\pi_1^y, \pi_2^y) = d(\pi_1, \pi_2) \leq 4$ , contradicting the assumption that  $\delta \geq 5$ .  $\square$

We now consider the case where  $C$  has minimum distance of 4, but also the size of  $Q$  is at least 3.

**Lemma 4.2.** *Let  $C$  be a code with  $\delta = 4$  and  $q \geq 3$ . Then  $C$  is fixed setwise by  $G_{\Gamma_1(C)}$ .*

*Proof.* Suppose  $C$  is not fixed setwise by  $G_{\Gamma_1(C)}$ . Then there exist  $\alpha \in C$  and  $y \in G_{\Gamma_1(C)}$  such that  $\alpha^y \notin C$ . Since  $q \geq 3$ , we let 0, 1 and 2 be distinct elements of  $Q$ . Let  $\pi \in \text{Pre}(\alpha, y)$  and  $\nu \in \Gamma_1(\alpha) \cap \Gamma_1(\pi)$ . By definition we know that  $d(\alpha, \pi) = 2$ . We also know, by Lemma 2.2, that  $G$  is transitive on triples  $(\alpha', \nu', \pi')$  with  $\alpha', \nu', \pi'$  vertices of  $H(m, q)$  such that  $d(\alpha', \pi') = 2$  and  $\nu' \in \Gamma_1(\alpha') \cap \Gamma_1(\pi')$ . So replacing  $C$  by an equivalent code if necessary, we may assume that  $\alpha = \mathbf{0}$ ,  $\nu = (2, 0, \dots, 0)$  and  $\pi = (2, 1, 0, \dots, 0)$ , and by Lemma 2.5, we can still assume that the minimum distance is  $\delta = 4$ . By part (iii) of Lemma 3.3,  $\Gamma_1(\pi) \subseteq \Gamma_1(C)$ . Thus  $\nu_2 = (1, 1, 0, \dots, 0) \in \Gamma_1(C)$ , and so  $\nu_2$  is the neighbour of a codeword  $\beta$ , say. It follows that  $\beta$  must have weight either 1, 2 or 3 and hence that  $d(\alpha, \beta) \leq 3 < \delta$ , which is a contradiction.  $\square$

**Lemma 4.3.** *Let  $C$  be a code with  $\delta \geq 3$  with  $q$  even and  $m$  odd. Then  $C$  is fixed setwise by  $G_{\Gamma_1(C)}$ .*

*Proof.* Suppose  $C$  is not fixed setwise by  $G_{\Gamma_1(C)}$ . Then by Lemma 3.3 there exist  $\alpha \in C$  and  $y \in G_{\Gamma_1(C)}$  such that  $2 \times |\text{Pre}(\alpha, y)| = m(q - 1)$ . Thus 2 divides either  $m$  or  $q - 1$ . If  $m$  is odd and  $q$  is even this is not possible.  $\square$

Lemmas 4.1–4.3 together yield a proof that at least one of (1), (2) or (3) of Theorem 1.2 holds.

## 5. INFINITE FAMILY OF BINARY CODES $C$

In this section we define a family of binary codes in  $H(m, 2)$  where  $m$  is even and at least 4. For a code  $C$  in this family, we prove that  $C$  has minimum distance  $\delta = 4$ , is  $G_{\Gamma_1(C)}$ -neighbour transitive, and that  $G_{\Gamma_1(C)}$  does not fix  $C$  setwise. That is, the final statement of Theorem 1.2 holds for this family of codes.

We can view the vertex set of  $H(m, 2)$  as the vector space  $\mathbb{F}_2^m$  of  $m$ -dimensional row vectors over  $\mathbb{F}_2$ . With this in mind, for each  $i \in M = \{1, \dots, m\}$ , we let  $e_i$  denote the vertex with 1 only in the  $i^{\text{th}}$  position. Furthermore, because the base group  $N \cong S_2^m$  of  $G = \text{Aut}(\Gamma) = N \rtimes L \cong S_2 \wr S_m$  is regular on the vertices of  $H(m, 2)$ , we may identify  $N$  with the group of translations of  $\mathbb{F}_2^m$ , and  $G$  with a subgroup of the affine group  $\text{AGL}(m, 2)$ . More precisely  $N$  consists of the translations  $\phi_\alpha$ , where  $\beta^{\phi_\alpha} = \beta + \alpha$  for  $\alpha, \beta \in \mathbb{F}_2^m$ , and if  $\mathbf{0}$  is the zero vector, then  $G = N \rtimes G_{\mathbf{0}}$  where  $G_{\mathbf{0}}$  is the group of permutation matrices in  $\text{GL}(m, 2)$ . For any subset  $S$  in  $\mathbb{F}_2^m$  we let  $\text{Perm}(S)$  denote the group of permutation matrices that fix  $S$  setwise.

Let  $m$  be even and at least 4. Then we can consider vectors in  $\mathbb{F}_2^m$  as 2-tuples of vectors from  $\mathbb{F}_2^{m/2}$ . In particular, for any vertex  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{F}_2^m$ , we can identify  $\alpha$  with the 2-tuple  $(\beta, \gamma)$  where  $\beta = (\alpha_1, \dots, \alpha_{m/2})$ ,  $\gamma = (\alpha_{m/2+1}, \dots, \alpha_m) \in \mathbb{F}_2^{m/2}$ . Given this, we define the following subsets of  $\mathbb{F}_2^m$ :

$$U = \{(\beta, \beta) \in \mathbb{F}_2^m : \beta \in \mathbb{F}_2^{m/2}\},$$

$$C = \{(\beta, \beta) \in \mathbb{F}_2^m : \text{wt}(\beta) \text{ is even in } \mathbb{F}_2^{m/2}\}.$$

It follows from the definitions that  $U$  and  $C$  are subspaces of  $\mathbb{F}_2^m$ , and thus are linear codes. The minimum weight of vectors in  $U$  and  $C$  is 2 and 4 respectively. Therefore the minimum distance of  $U$  and  $C$  is  $\delta_U = 2$  and  $\delta_C = 4$  respectively. Also, it is straight forward to deduce that

$$\Gamma_1(U) = \{(\beta, \gamma) \in \mathbb{F}_2^m : d(\beta, \gamma) = 1\},$$

where  $d(\beta, \gamma)$  is the Hamming distance between  $\beta$  and  $\gamma$  in  $H(m/2, 2)$ . We now show that the neighbour sets of  $U$  and  $C$  coincide.

**Lemma 5.1.**  $\Gamma_1(U) = \Gamma_1(C)$ .

*Proof.* Because  $\delta_U = 2$  we have that  $\Gamma_1(U) = \cup_{(\beta, \beta) \in U} \Gamma_1((\beta, \beta))$ , and so  $\Gamma_1(C) \subseteq \Gamma_1(U)$ . Conversely, suppose  $(\beta, \gamma) \in \Gamma_1(U)$ . Then  $d(\beta, \gamma) = 1$  in  $H(m/2, 2)$ , and so  $\text{wt}(\beta)$  and  $\text{wt}(\gamma)$  have opposite parity. From this we conclude that either  $(\beta, \beta) \in C$  or  $(\gamma, \gamma) \in C$ . In either case it follows that  $(\beta, \gamma) \in \Gamma_1(C)$ .  $\square$

Let  $J_1 = \{1, \dots, \frac{m}{2}\}$  and  $J_2 = \{\frac{m}{2} + 1, \dots, m\}$ , and consider the partition  $\mathcal{J} = \{J_1, J_2\}$  of  $M$ . Let  $H$  be the stabiliser of  $\mathcal{J}$  in  $S_m$ . Then  $H \cong S_{m/2} \text{ wr } S_2$ , and a typical element of  $H$  is of the form  $(\sigma_1, \sigma_2)\hat{\sigma}$  where  $\sigma_1, \sigma_2 \in S_{\frac{m}{2}}$  and  $\hat{\sigma} \in S_2$ . Let  $K = \{(\sigma_1, \sigma_2)\hat{\sigma} \in H : \sigma_1 = \sigma_2\}$ . Then  $K \cong S_{\frac{m}{2}} \times S_2$ , a transitive subgroup of  $S_m$  and we can identify  $K$  with a subgroup of permutation matrices in  $\text{GL}(m, 2)$ . As such, for  $y = (\sigma, \sigma)\hat{\sigma} \in K$  we have that  $(\beta, \beta)^y = (\beta^\sigma, \beta^\sigma)$  for all  $(\beta, \beta) \in U$ . Therefore  $K$  stabilises  $U$ . Furthermore, because permutation matrices in  $\text{GL}(m/2, 2)$  preserve weights of vectors in  $\mathbb{F}_2^{m/2}$ , it follows that  $K$  stabilises  $C$  also.

**Lemma 5.2.**  $C$  is  $\text{Aut}(C)$ -neighbour transitive.

*Proof.* By Lemma 2.3,  $\text{Aut}(C)$  fixes  $\Gamma_1(C)$  setwise. Let  $\nu_1, \nu_2 \in \Gamma_1(C)$ . Then there exist  $\alpha_i \in C$  such that  $\nu_i \in \Gamma_1(\alpha_i)$  for  $i = 1, 2$ . It follows that the translation  $\phi_{\alpha_i}$  maps  $\alpha_i$  to  $\mathbf{0}$  for  $i = 1, 2$ . Because adjacency is preserved by automorphisms of  $\Gamma$ , we have that  $\nu_i^{\phi_{\alpha_i}} \in \Gamma_1(\mathbf{0})$  for  $i = 1, 2$ . Therefore there exists  $s, t \in M$  such that  $\nu_1^{\phi_{\alpha_1}} = e_s$  and  $\nu_2^{\phi_{\alpha_2}} = e_t$ . Since  $K \leq \text{Aut}(C)$  acts transitively on  $M$ , and because permutation matrices preserve weight, it follows that there exists  $\sigma \in K$  such that  $e_s^\sigma = e_t$ . Hence  $\nu_1^{\phi_{\alpha_1} \sigma \phi_{\alpha_2}} = \nu_2$ . Finally, because  $\alpha_i \in C$  and  $C$  is linear, we have  $\phi_i \in \text{Aut}(C)$ , and also  $\sigma \in K \leq \text{Aut}(C)$ , so  $\phi_1 \sigma \phi_2 \in \text{Aut}(C)$ .  $\square$

Since  $U$  is a binary linear code, we can conclude from [6, Lemma 3.1] that  $\text{Aut}(U) = N_U \rtimes \text{Perm}(U)$ , where  $N_U$  is the group of translations generated by  $U$ . By Lemma 2.3,  $\text{Aut}(U)$  fixes  $\Gamma_1(U)$  setwise, and by Lemma 5.1,  $\Gamma_1(U) = \Gamma_1(C)$ . Therefore  $\text{Aut}(U) \leq G_{\Gamma_1(C)}$ . Since  $N_U$  does not fix  $C$  setwise it follows that  $G_{\Gamma_1(C)}$  does not fix  $C$  setwise. Furthermore, Lemma 2.3 and Lemma 5.2 imply that  $G_{\Gamma_1(C)}$  acts transitively on  $\Gamma_1(C)$ . Hence  $C$  is  $G_{\Gamma_1(C)}$ -neighbour transitive but is not fixed setwise by  $G_{\Gamma_1(C)}$ . Thus we have proved the final statement of Theorem 1.2.

**Remark 5.3.** Equivalent codes to the ones described in this section can be found in [4, Chapter 3], in which the automorphism groups of these codes are given. From this we can conclude that for  $m \geq 6$ ,  $\text{Aut}(U) = N_U \rtimes K'$  and  $\text{Aut}(C) = N_C \rtimes K'$  where  $K' \cong S_2 \text{ wr } S_{m/2}$ . Furthermore,  $G_{\Gamma_1(C)} = N_U \rtimes K'$ .



The case where  $m = 4$  is an exception. In this case  $C = \{\mathbf{0}, \mathbf{1}\}$  (where  $\mathbf{1} = (1, 1, 1, 1)$ ) and  $\text{Aut}(C) = N_C \rtimes G_{\mathbf{0}}$ . However,  $\text{Aut}(U) = N_U \rtimes K'$  where  $K' \cong D_8$  and  $G_{\Gamma_1(C)} = N_W \rtimes G_{\mathbf{0}}$  where  $W$  is the subspace of  $\mathbb{F}_2^4$  consisting of all even weight vectors. Again it follows that  $C$  is  $G_{\Gamma_1(C)}$ -neighbour transitive but  $G_{\Gamma_1(C)}$  does not fix  $C$  setwise. Interesting, however, because  $\Gamma_1(C) = \Gamma_1(U)$ , in this case we also have that  $U$  is  $G_{\Gamma_1(U)}$ -neighbour transitive but  $G_{\Gamma_1(U)}$  does not fix  $U$  setwise.

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